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Weak asymptotic abelianess for Galilei invariant time evolution

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Abstract
We extend the result of weak asymptotic abelianess for Galilei invariant time evolution from the tracial state to a set of time-invariant states which are uniformly clustering with respect to space translation.

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1. Introduction

A realistic physical system should be described in the framework of relativistic quantum field theory [1]. But in this framework the available concrete results are to a large extent restricted only to ground states. The construction of temperature states needs some further assumptions like nuclearity. But even with this assumption it is not clear how the choice of the local states determines in the limit the global state [2]. If we therefore want to describe and understand effects of macroscopic systems that justify a thermodynamical limit, but with velocities by far smaller than the velocity of light, we have turned our interest to simplified models. Here we can still want to stay on a more abstract level that covers a wide area of phenomena. To do so we start with a C^* algebra on which time and space translations are assumed to act as automorphisms [3]. This includes especially models on lattices, which by construction admit space translations as automorphisms, and where with appropriate assumptions on the interaction by analytic perturbation methods time evolution can be shown to exist as an automorphism. These perturbation methods fail for continuous systems even with short-range interaction, because lack of thermodynamic stability for at least one sign of the interaction hinders perturbation theory from converging. Therefore in [4] we proposed an interaction that does not only vanish when the particles are far apart but also when their relative velocity is very large so that it becomes impossible that an infinite amount of energy is exchanged between the particles. This enabled us to construct, again via perturbation theory, a time automorphism on the standard C^* algebra of fermions built by creation and annihilation operators.

The advantage of this construction was the fact that time evolution, space translation and boost are related by Galilei invariance, as it should be in the nonrelativistic limit of quantum field theory, a fact which has no counterpart in lattice systems. Galilei invariance connects space translation and time translation sufficiently so that it can be proven that in the tracial state where all automorphisms are unitarily implementable time evolution inherits the continuous spectrum of space translation and therefore also weak asymptotic abelianess. This was proven in [5]. Weak asymptotic abelianess is a property that refers to a special representation and implies that two point correlations in time vanish at infinity. It is natural to hope that this property holds for all extremal time-invariant states, not just for the tracial state. In fact we tried and succeeded in transferring weak asymptotic abelianess in the tracial state into a weaker property on the level of the algebra, not just restricted to special states, namely mixing.

Mixing is a property familiar in classical ergodic systems. Transferred to quantum systems it reflects the fact that operators do not remain local. Already the spreading of wavefunctions suffices to produce delocalization. Additional interaction should only improve this delocalization. Mixing in the spirit of ergodic theory is thus a minimal requirement to understand thermodynamics. However its consequences on correlation functions are not under control. Weak asymptotic abelianess in all invariant states however would have far-reaching consequences and is the starting point for many abstract considerations [3], especially concerning the decomposition into extremal invariant states, or return to equilibrium under small perturbations. Of course strong asymptotic abelianess respectively norm asymptotic abelianess, which does not refer to a state, might be even more desirable. Since it holds for the free evolution at least on the observable algebra, it might also seem probable. But here we have the counterexample of the *XY*-model [6, 7] which is definitely not norm asymptotically Abelian. Though this is a model on a lattice it reflects the effect of interactions between neighboring points, and it is plausible that it does not behave worse than a continuous system. Therefore weak asymptotic abelianess is the best we can hope to prove in order to gain good control on thermodynamic systems. Surprisingly enough it turns out that from the view point of thermodynamics it is even more powerful than norm asymptotic abelianess: the example of the Price–Powers shift indicates [8, 9] that weak asymptotic abelianess is as powerful to describe extremal invariant states and to show why on the macroscopic level we can explain the phenomena by classical mechanics, but it can be by far more demanding in the search for invariant states: for almost all price-powers shifts the only invariant state is the tracial state.

Therefore we should not expect that norm asymptotic abelianess could hold in general for Galilei invariant interactions, but weak asymptotic abelianess should hold in all invariant states to guarantee good thermodynamical behaviour. In this paper we will succeed in showing that it holds to arbitrary good precision for a set of invariant states under the assumption of some uniformity in space clustering. As a by-product it follows that for these states space translation cannot be broken in time-invariant states.

2. The model

We start with the C^* algebra [3] built by the creation and annihilation operators $a(f), a^\dagger(g)$ for which $[a(f), a^\dagger(g)]_+ = \int \mathrm{d}x\, f(x)\bar{g}(x)$. In general we consider time evolution given by a Hamiltonian

$$H = \frac{1}{2m}\int \mathrm{d}x \nabla_x a^\dagger(x)\nabla_x a(x) + \int \mathrm{d}x\, \mathrm{d}y a^\dagger(x)a^\dagger(y)v(x-y)a(y)a(x).$$

This defines a time evolution in Fock space but does not allow us to construct a time evolution as automorphism group on the C^* algebra. Such an automorphism group is in general constructed as a perturbation series over the potential, and since at least for one sign of the potential the Hamiltonian is not bounded from below by $-cN$, particles can accelerate arbitrarily and we cannot expect that time evolution exists for this sign. The generally used estimates on the convergence of perturbation theory do not depend on the sign and must therefore fail. If however the potential is such that $H > -cN$ the kinetic energy cannot increase by an accumulation of the particles and we can expect that the individual particles do not become too fast. Therefore it is plausible that cutting the interaction for particles with high velocity should not change physics substantially. This was done in [4] by considering a Hamiltonian

$$H = \frac{1}{2m}\int \mathrm{d}x \nabla_x a^\dagger(x)\nabla_x a(x) + \int \mathrm{d}p\,\mathrm{d}p'\mathrm{d}q\,\mathrm{d}q' a^\dagger_{pq} a^\dagger_{p'q'} v(p-p', q-q')) a_{p'q'} a_{pq}. \qquad (1)$$

Here $a_{pq} = a(W(p,q)f)$ are annihilation operators smeared with an f that is translated by the Weyl operators. As a concrete example we took in [4] f as a Gauss function so that with appropriate normalization in three dimensions

$$a_{pq} = \pi^{-\frac{3}{4}} \int \mathrm{d}^3 x\, \mathrm{e}^{-\frac{(q-x)^2}{2} + \mathrm{i}px}\, a(x).$$

With such an interaction stability is guaranteed

$$V \geqslant -N\|v\|_1, \qquad \|v\|_1 = \frac{1}{(2\pi)^3}\int \mathrm{d}^3x\,\mathrm{d}^3p |v(p,x)|.$$

The effect of the potential on the time evolution can now be calculated in the same way as time evolution is calculated for lattice systems, and we obtain a time evolution as an automorphism group τ_t implemented by the above Hamiltonian.

3. Galilei invariance

We consider again our C^* algebra built by creation and annihilation operators. On this algebra we have the following automorphism groups:

space translation $\sigma_x a(f(y)) = a(f(x+y))$
boost $\gamma_b a(f(y)) = a(\mathrm{e}^{\mathrm{i}by} f(y))$
gauge automorphism $\nu_\alpha a(f) = \mathrm{e}^{\mathrm{i}\alpha} a(f)$
time automorphism τ_t, which we assume to exist.

These automorphisms satisfy the following commutation relations:

$$\sigma_x \circ \nu_\alpha = \nu_\alpha \circ \sigma_x, \qquad \gamma_b \circ \nu_\alpha = \nu_\alpha \circ \gamma_b, \qquad \gamma_b \circ \sigma_x = \sigma_x \circ \gamma_b \circ \nu_{bx}$$

Therefore space translations and boost commute on the gauge invariant subalgebra and this subalgebra is stable under space translation and boost.

Definition. *A time evolution is called gauge and Galilei invariant (for simplicity we fix* $m = 1$*) if the following relations hold:*

$$\begin{aligned} &\tau_t \circ \nu_\alpha = \nu_\alpha \circ \tau_t \qquad \tau_t \circ \sigma_x = \sigma_x \circ \tau_t \\ &\tau_t \circ \gamma_b = \gamma_b \circ \tau_t \circ \sigma_{bt} \circ \nu_{-b^2t/2}. \end{aligned} \qquad (2)$$

Lemma. *The free time evolution* $\tau_t^0 a(\tilde f(p)) = a\big(^{-\mathrm{i}p^2t/2} \tilde f(p)\big)$ *is Galilei and gauge invariant. This can easily be seen by its action on an annihilation operator.*

Lemma. *The time evolution defined by Hamiltonian (1) is gauge and Galilei invariant.*

Proof. Since the potential was constructed in such a way that $\sigma_x[V, A] = [V, \sigma_x A]$, $\gamma_b[V, A] = [V, \gamma_b A]$, the time evolution inherits the invariance from the free time evolution, as is shown in detail in [4]. □

4. Weak clustering

For convenience we restrict our considerations to the gauge invariant C^* algebra. Extension to the total algebra makes it necessary that we have to include the gauge automorphism, but does not lead to different results. We start with a state ω that is extremal invariant under space translations and invariant under time translations. We can obtain such a state e.g. if we take the invariant mean over space translations of a state that is a temperature state and therefore satisfies the KMS condition (Kubo–Martin–Schwinger) [3, 1] with respect to the time evolution. Therefore we do not talk about an empty set. If we consider $\omega_b = \omega \circ \gamma_b$ this state is again extremal invariant under space translation and invariant under time translation. We smear over the boost and obtain

$$\omega_f(A) = \int \mathrm{d}b\, \omega(\gamma_b A) f(b)$$

where we take $f(b)$ to be a positive function with $\int \mathrm{d}b f(b) = 1$. We can assume that it is proportional to $\mathrm{e}^{-\lambda(b-c)^2}$ or a positive linear combination of such functions. ω_f can be decomposed into the extremal space translation invariant states $\omega_b = \omega \circ \gamma_b$ and this decomposition is unique: taking into account that space translations are norm- and therefore also weakly asymptotically Abelian it follows that the decomposition of ω_f with respect to db is coarser than the central decomposition or agrees with the central decomposition. Therefore in the GNS (Gelfand–Naimark Segal) representation [3, 1] corresponding to ω_f respectively in its central decomposition the weak limit with respect to space translations $m_f(A)$ exists and belongs to the center. We can write it therefore in the following way:

$$w - \lim_{x\to\infty} \pi_f(\sigma_x A) = w - \lim_{x\to\infty} \bigoplus_f \pi_b(A) = m_f(A) = \bigoplus_f \omega_b(A) 1_b \tag{3}$$

exists and belongs to the center. We want to prove that also

$$w - \lim_{t\to\infty} \pi_f(\tau_t A) = m_f(A) = \bigoplus_f \omega_b(A) 1_b \tag{4}$$

in the corresponding representation for a dense set of operators A. Weak asymptotic abelianess follows if we can prove: for a dense set of operators A, B and $\forall \epsilon > 0$ there exists $t_0(A, B, \epsilon)$ such that for $t > t_0$

$$|\langle \Omega_f | \pi_f(A)(\pi_f(\tau_t B) - m_f(B)) | \Omega_f \rangle| < \epsilon. \tag{5}$$

We take $f(b)$ to be a convolution and estimate

$$\begin{aligned}\langle \Omega_f | \pi_f(A)(\pi_f(\tau_t B) - m_f(B)) | \Omega_f \rangle &= \int \mathrm{d}b \langle \Omega_b | \pi_b(A)(\pi_b(\tau_t B) - \omega_b(B) | \Omega_b \rangle f(b) \\ &= \int \mathrm{d}b\, \mathrm{d}b' \langle \Omega_{b+b'} | \pi_{b+b'}(A)(\pi_{b+b'}(\tau_t B) - \omega_{b+b'}(B)) | \Omega_{b+b'} \rangle f_\mu(b') g_\mu(b).\end{aligned}$$

With $g_\mu(b')$ proportional to $\mathrm{e}^{-\frac{b'^2}{\mu}}$ and $\int \mathrm{d}b g_\mu(b) = 1$ we have with our assumption on f that $f_\mu(b) > 0$ and $\lim_{\mu\to 0} f_\mu(b) = f(b)$. We choose A and B continuous with respect to the boost such that

$$\|\gamma_b A - A\| < \epsilon_1 \qquad \|\gamma_b B - B\| < \epsilon_1 \qquad \forall |b| < b_0 \tag{6}$$

where b_0 depends on A, B, ϵ_1. These operators are dense in the C^* algebra. Accordingly we choose μ such that $\int_{|b|>b_0} \mathrm{d}b g_\mu(b) < \epsilon_1$. This allows us to continue

$$= \int \mathrm{d}b' f_\mu(b') \int_{|b|<b_0} \mathrm{d}b \langle \Omega_{b'} | \pi_{b'}(\gamma_b A)(\pi_{b'}(\gamma_b \tau_t B) - \omega_{b'}(B)) | \Omega_{b'} \rangle g_\mu(b) + O(\epsilon_1).$$

Now the Cauchy-Schwartz inequality allows us to estimate up to order ϵ_1, again using the continuity of $\gamma_b A$

$$\begin{aligned} &\left| \int \mathrm{d}b' f_\mu(b') \int_{|b|<b_0} \mathrm{d}b \langle \Omega_{b'} | \pi_{b'}(\gamma_b A)(\pi_{b'}(\gamma_b \tau_t B) - \omega_{b'}(B)) | \Omega_{b'} \rangle g_\mu(b) \right| \\ &\quad \leqslant O(\epsilon_1) + \int \mathrm{d}b' f_\mu(b') |\langle \Omega_{b'} | (\pi_{b'}(A)(\pi_{b'}(A^\dagger) | \Omega_{b'} \rangle|^{\frac{1}{2}} \\ &\quad \times \left| \int_{|b|<b_0, |\bar{b}|<b_0} \mathrm{d}b \, \mathrm{d}\bar{b} \langle \Omega_{b'} | (\pi_{b'}(\gamma_b \tau_t B^\dagger) \right. \\ &\quad \left. - \omega_{b'}(B^\dagger))(\pi_{b'}(\gamma_{\bar{b}} \tau_t B) - \omega_{b'}(B)) | \Omega_{b'} \rangle g_\mu(b) g_\mu(\bar{b}) \right|^{\frac{1}{2}}. \end{aligned} \tag{7}$$

Using the Galilei invariance of τ_t (2) the last line can be replaced by

$$\begin{aligned} &\left| \int_{|b|<b_0, |\bar{b}|<b_0} \mathrm{d}b \, \mathrm{d}\bar{b} \langle \Omega_{b'} | (\pi_{b'}(\tau_t \sigma_{bt} \gamma_b B^\dagger) - \omega_{b'}(B^\dagger))(\pi_{b'}(\tau_t \sigma_{\bar{b}t} \gamma_{\bar{b}} B) \right. \\ &\quad \left. - \omega_{b'}(B)) | \Omega_{b'} \rangle g_\mu(b) g_\mu(\bar{b}) \right|^{\frac{1}{2}} \\ &\quad = \left| \int_{|b|<b_0, |\bar{b}|<b_0} \mathrm{d}b \, \mathrm{d}\bar{b} \langle \Omega_{b'} | (\pi_{b'}(\gamma_b B^\dagger) - \omega_{b'}(B^\dagger))(\pi_{b'}(\sigma_{(\bar{b}-b)t} \gamma_{\bar{b}} B) \right. \\ &\quad \left. - \omega_{b'}(B)) | \Omega_{b'} \rangle g_\mu(b) g_\mu(\bar{b}) \right|^{\frac{1}{2}}. \end{aligned} \tag{8}$$

where we have used that the state $\omega_{b'}$ is invariant under space and time translations. Now however we need an additional assumption on the state: we know that $\omega_{b'}$ is an extremal invariant state under space translation and therefore clustering in space. Since we integrate over b' we have to assume that this clustering is uniform over the integration region of b'. Using that $|b - \bar{b}| \leqslant 2b_0$ this assumption implies that the integral becomes small up to ϵ $\forall |t| \gg \frac{1}{2b_0}$. b_0 determines how we have to choose μ in order to satisfy (6), namely μ has to be at least of the order b_0^2. Therefore (8) becomes small for t of the order $|t| \gg \frac{1}{\sqrt{\mu}}$. Collecting the estimates we have shown: for a given A in a dense set and a given ϵ we have to choose μ small, so that we can use the continuity with respect to the boost in (6) together with the fact that, as defined in (3), $m_{f_\mu}(A)$ approaches weakly $m_f(A)$ for $\mu \to 0$. But we can choose it finite so that $t \gg \mu^{-1/2}$ can still be satisfied. For these $t > t_0(A, B, \epsilon)$

$$|\langle \Omega_f | \pi_f(A)(\pi_f(\tau_t B) - m_f(b)) | \Omega_f \rangle| < \epsilon.$$

This fact can be rephrased in the following theorem:

Theorem. *Let ω be a state which is extremely space translation invariant and time-invariant, and ω_f the corresponding state smeared with the boost. Assume further that ω and therefore also ω_f are cyclic and separating for $\pi_\omega(\mathcal{A})''$. Assume that the extremal invariant states cluster uniformly in space. Then for all A that are strongly continuous with respect to the boost $w - lim_{t\to\infty}\pi_f(\tau_t A) = w - lim_{x\to\infty}\pi_f(\sigma_x A) = m_f(A)$.*

Note that the set of states ω_f constructed in the above way is weakly dense among the sets of states that are both time and space translation invariant. What is missing so far is the control on the clustering with respect to space translations.

From this limiting behaviour we can learn more about the time evolution.

Lemma. *Let U_t and V_x implement the time evolution and the space translations in the GNS representation of the state ω_f, respectively. Then the projections P_t and Q_x onto the time-invariant, respectively, space invariant vectors obtained as*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\,U_t = P_t, \qquad \lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{X}\mathrm{d}x\,V_x = Q_x$$

exist, they coincide and satisfy that the invariant operators in the weak closure of the algebra belong to the center

$$P_t\pi_f(\mathcal{A})''P_t = Q_x\pi_f(\mathcal{A})''Q_x \subset \mathcal{Z}. \tag{9}$$

Proof. Since the projections are bounded operators it is sufficient to control their construction on a dense set, and this set is given by the operators smooth under the boost. For Q_x we know that space translations are strongly asymptotically Abelian which implies (9).Therefore the decomposition into extremal space translation invariant states is equal or coarser than the central decomposition. □

Note that we could control the time convergence to a central operator only for a dense set of operators. But (9) suffices to guarantee [3, 10] that the decomposition of ω_f into time-invariant states is unique and coincides with the decomposition into space translation invariant states and thus with the decomposition given by f. But this makes it impossible that space translation is broken in a time-invariant state. Especially this implies that extremal KMS states which therefore have trivial center are automatically also space translation invariant, provided clustering properties with respect to space translation do not change under the boost. Note, however, that this does not exclude a crystallin structure, since in a Galilei invariant theory both electrons and nuclei are included.

5. Broken time symmetry

In the last section we started with a time-invariant state and assumed it to be extremal space invariant. It turned out that as a consequence it is also extremely time-invariant and space symmetry cannot be broken in a time-invariant state. However we were not able to argue on the basis of Galilei invariance that time invariance also cannot be broken in space translation invariant states. We start now with an extremal space invariant state that is only invariant under τ_n so that $\omega_\alpha = \omega\circ\tau_\alpha \neq \omega, \forall\, 0 < \alpha < 1$. We can proceed in the same way as before constructing $\omega_{\alpha,b'}$ respectively $\omega_{\alpha,f}$. For every fixed α we proceed as before only replacing $\lim_t$ by $\lim_n$. Again

$$\lim_x \pi_{\alpha,f}(\sigma_x A) = \lim_n \pi_{\alpha,f}(\tau_n A),$$

but the dependence on α does not disappear so that states periodic in time seem to be possible on the basis of Galilei invariance.

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